MSc Computer Games and Entertainment

Maths & Graphics Unit 2011/12

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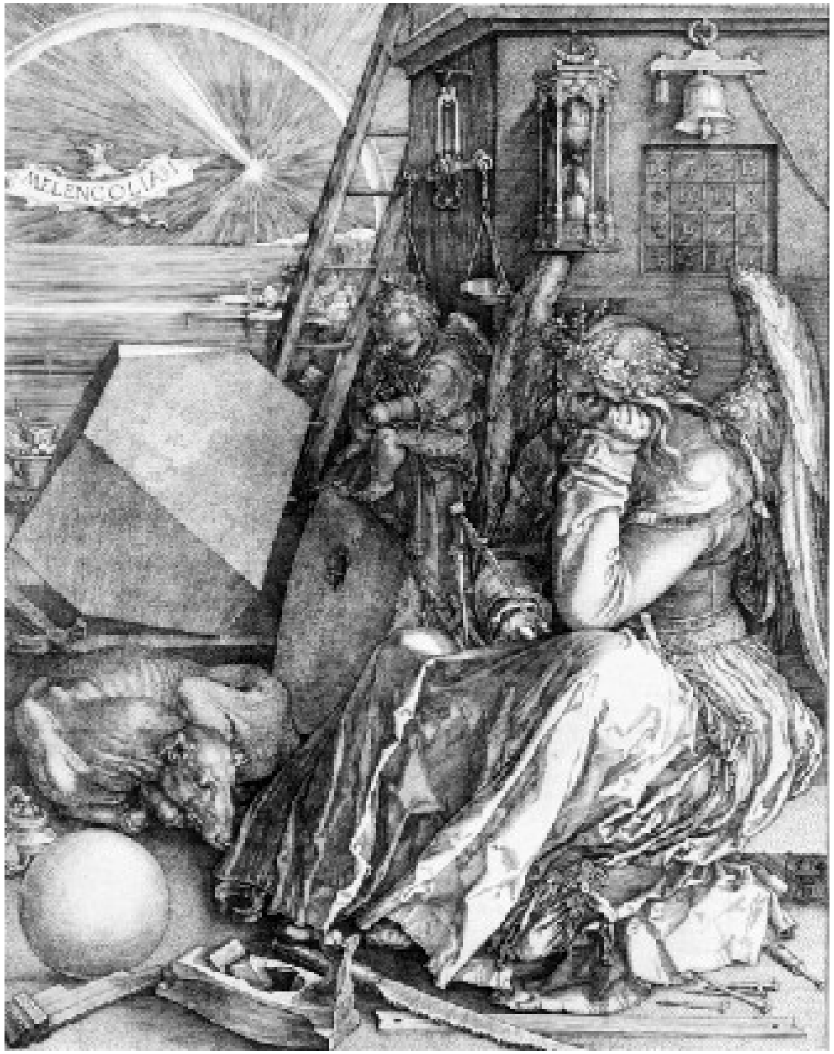
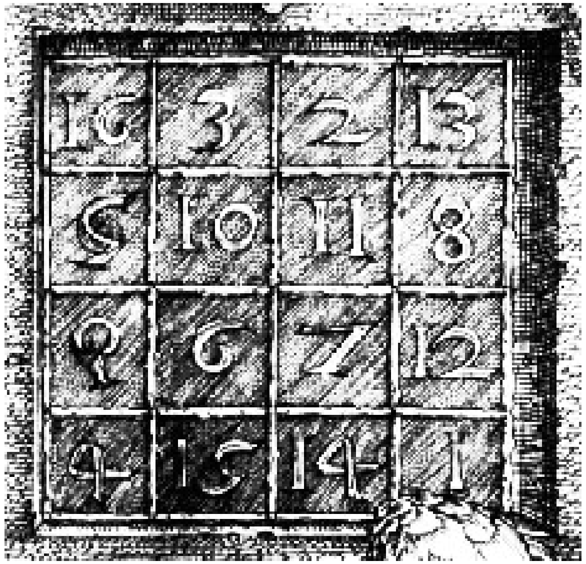
Matrices

$$\mathbf{A}_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$



Renaissance engraving Melencolia I by the German
artist and amateur mathematician Albrecht Durer.



Melencolia I is filled with mathematical symbolism, and if you look carefully, you will see a matrix in the upper right corner.

This matrix is known as a magic square and was believed by many in Durer's time to have genuinely magical properties.

It does turn out to have some fascinating characteristics worth exploring.

More On Matrices

- **Definition of a Matrix**
  - Rectangular array of real numbers
  - m rows by n columns
  - Named using capital letters
  - First subscript is row, second subscript is column

More On Matrices

- **Terminology**
  - A matrix with *m* rows and *n* columns is called a matrix of order *m* x *n*.
  - A square matrix is a matrix with an equal number of rows and columns. Since the number of rows and columns are the same, it is said to have order *n*.
  - The main diagonal of a square matrix are the elements from the upper left to the lower right of the matrix.
  - A row matrix is a matrix that has only one row.
  - A column matrix is a matrix that has only one column.
  - A matrix with only one row or one column is called a vector.

More On Matrices

- **Numbers**
  - A single number in matrix notation is called a scalar.

    It can be looked at as a number, or as a 1 x 1 matrix, or as a one element row or column.

More On Matrices

- **Rows**
  - A row (also called a row vector) is just an ordered collection of elements. For example

    `[ a b c ]`

  - is a row.

More On Matrices

- **Rows**

  - If you have two rows of the same length, you can add the rows by adding the corresponding elements in each row.

    For example, the row:

    [d e f] + [g h i] = [d+g e+h f+i]

More On Matrices

- **Rows**
  - One can multiply a row by a scalar (number).

    For example:

    ```
    2 0 [ a  b  c ] = [ 2a  2b  2c ]
    ```

  - A row may have any number of elements, from one upwards.
  - If Z is a row, Z(i) means the i'th element of that row.

More On Matrices

- **Columns**
  - A column (also called a column vector) is just like a row, except it is arranged vertically.
  - For example:

```
[ a ]
[ b ]
[ c ]
```

More On Matrices

- **Columns**

  – Columns can be added, or multiplied by a scalar (number) the same way that rows can:

  For example:

```
[ a ]   [ g ]   [ a+g ]
[ b ] + [ h ] = [ b+h ]
[ c ]   [ i ]   [ c+i ]

    [ a ]     [ 2a ]
2 0 [ b ]  =  [ 2b ]
    [ c ]     [ 2c ]
```

More On Matrices

- **Columns**
  - A column may have any number of elements, from one on up.
  - If Y is a column, then Y(i) means the i'th element of the column, counting from the top.

More On Matrices

- A diagonal matrix is one that has non-zero elements only on the diagonal.

```
[ 0 * * * ]
[ * 0 * * ]
[ * * 0 * ]
[ * * * 0 ]
```

More On Matrices

- A block diagonal matrix is like a diagonal matrix, except that elements exist in the positions arranged as blocks.

```
[ 0 * * * * * * * ]
[ * 0 0 * * * * * ]
[ * 0 0 * * * * * ]
[ * * * 0 0 0 * * ]
[ * * * 0 0 0 * * ]
[ * * * 0 0 0 * * ]
[ * * * * * * 0 0 ]
[ * * * * * * 0 0 ]
```

More On Matrices

- A Band Matrix has numbers near the diagonal of the matrix, and nowhere else. The width of the band is called the band width of the matrix.

```
[ 0 0 0 * * * * * ]
[ 0 0 0 0 * * * * ]
[ 0 0 0 0 0 * * * ]
[ * 0 0 0 0 0 * * ]
[ * * 0 0 0 0 0 * ]
[ * * * 0 0 0 0 0 ]
[ * * * * 0 0 0 0 ]
[ * * * * * 0 0 0 ]
```

More On Matrices

- A matrix is called sparse if most of the elements in it are zero.

```
[ * * * 0 * * * * ]
[ * 0 * * * * 0 * ]
[ * * * * 0 * * * ]
[ * * 0 * * * * * ]
[ 0 * * * * * * * ]
[ * * * * * * * 0 ]
[ * * 0 * * 0 * * ]
[ * * * 0 * * * * ]
```

More On Matrices

- A matrix is called sparse if most of the elements in it are zero.

```
[ * * * 0 * * * * ]
[ * 0 * * * * 0 * ]
[ * * * * 0 * * * ]
[ * * 0 * * * * * ]
[ 0 * * * * * * * ]
[ * * * * * * * 0 ]
[ * * 0 * * 0 * * ]
[ * * * 0 * * * * ]
```

More On Matrices

- A matrix is called dense if most of the elements in it are NOT zero.

```
[ * * * 0 * * * * ]
[ * 0 * * * * 0 * ]
[ * * * * 0 * * * ]
[ * * 0 * * * * * ]
[ 0 * * * * * * * ]
[ * * * * * * * 0 ]
[ * * 0 * * 0 * * ]
[ * * * 0 * * * * ]
```

More On Matrices

- The transpose of a matrix "N" (Written N') is just a matrix "P" such that N(i,j) = P(j,i).

- For example (where the matrix has been reflected across the diagonal):

```
Transpose ( [ a b c ] ) = [ a d ]
            [ d e f ]     [ b e ]
                          [ c f ]
```

- Or with ' notation:

```
If A = [ a b c ]        then A' = [ a d ]
       [ d e f ]                  [ b e ]
                                  [ c  f ]
```

[c f]

More On Matrices

- If a matrix M has the property that:
- $M'M = I$
- Then this matrix is called an **Orthogonal matrix**

More On Matrices

- The transpose of a product (AB)' is the product of the individual transposes in reverse order. (B'A')

- The transpose of a row is a column, and the transpose of a column is a row.

  For example:

```
[ a b c]`  =    [ a ]
                [ b ]
                [ c ]
```

More On Matrices

- It is usually easier to write column matrices using transposes as:

```
[a b c]'
```

Than as

```
[a]
[b]
[c]
```

More On Matrices

- If a matrix is equal to its own transpose, it is called a symmetric matrix.
- The transpose of a number is just a number.
- (AB)' = B'A' in general. (A transpose of a product is the product of the transposes in reverse order)

More On Matrices

- A symmetric matrix is a square matrix equal to it's transpose.

  For example:

```
[ a b c d ]
[ b e f g ]
[ c f h i ]
[ d g i j ]
```

More On Matrices

- A symmetric matrix is a square matrix equal to it's transpose.

  For example:

```
[ a b c d ]
[ b e f g ]
[ c f h i ]
[ d g i j ]
```

- A symmetric matrix has the property that A = A'.

Operations with Matrices

$$\mathbf{A}_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

Operations with Matrices

- **Equality**
  - Two matrices are equal if and only if:
    - The order of the matrices are the same
    - The corresponding elements of the matrices are the same

$$\mathbf{A}_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

Operations with Matrices

- **Addition**
  - Order of the matrices must be the same
  - Add corresponding elements together
  - Matrix addition is commutative
  - Matrix addition is associative

$$\mathbf{A}_{m\times n}=\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

Operations with Matrices

- **Subtraction**

  – The order of the matrices must be the same

  – Subtract corresponding elements

  – Matrix subtraction is not commutative (neither is subtraction of real numbers)

  – Matrix subtraction is not associative (neither is subtraction of real numbers)

$$\mathbf{A}_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

Operations with Matrices

- **Scalar Multiplication**
  - A scalar is a number, not a matrix
    - The matrix can be any order
    - Multiply all elements in the matrix by the scalar
    - Scalar multiplication is commutative
    - Scalar multiplication is associative

$$\mathbf{A}_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

Operations with Matrices

- **Zero Matrix**
  - Matrix of any order
  - Consists of all zeros
  - Denoted by capital O
  - Additive Identity for matrices
  - Any matrix plus the zero matrix is the original matrix

$$\mathbf{A}_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

Operations with Matrices

- **Matrix Multiplication**

$$A_{m\times n} \times B_{n\times p} = C_{m\times p}$$

  - The number of columns in the first matrix must be equal to the number of rows in the second matrix. That is, the inner dimensions must be the same.

  - The order of the product is the number of rows in the first matrix by the number of columns in the second matrix. That is, the dimensions of the product are the outer dimensions.

$$\mathbf{A}_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

Operations with Matrices

- **Matrix Multiplication**
  - Since the number of columns in the first matrix is equal to the number of rows in the second matrix, you can pair up entries.
  - Each element in row *i* from the first matrix is paired up with an element in column *j* from the second matrix.
  - The element in row *i*, column *j*, of the product is formed by multiplying these paired elements and summing them.
  - Each element in the product is the sum of the products of the elements from row *i* of the first matrix and column *j* of the second matrix.
  - There will be *n* products which are summed for each element in the product.

$$\mathbf{A}_{m \times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

Operations with Matrices

- **Matrix Multiplication**
  - Consider the product of a 2×3 matrix and a 3×4 matrix.
  - The multiplication is defined because the inner dimensions (3) are the same. The product will be a 2×4 matrix, the outer dimensions

$$\begin{bmatrix} 1 & -2 & 3 \\ 4 & 5 & -2 \end{bmatrix}\begin{bmatrix} 1 & -8 & 4 & -3 \\ -3 & 6 & 7 & 2 \\ 6 & 5 & -1 & 4 \end{bmatrix}$$

Operations with Matrices

- **Matrix Multiplication**

  – Consider the product of a 2×3 matrix and a 3×4 matrix.

  – The multiplication is defined because the inner dimensions (3) are the same. The product will be a 2×4 matrix, the outer dimensions.

$$\begin{bmatrix} 1 & -2 & 3 \\ 4 & 5 & -2 \end{bmatrix} \begin{bmatrix} 1 & -8 & 4 & -3 \\ -3 & 6 & 7 & 2 \\ 6 & 5 & -1 & 4 \end{bmatrix}$$

  – Since there are three columns in the first matrix and three rows in the second matrix (the inner dimensions which must be the same), each element in the product will be the sum of three products.

Operations with Matrices

- **Matrix Multiplication**
  - **Row 1, Column 1**
  - To find the element in row 1, column 1 of the product, we will take row 1 from the first matrix and column 1 from the second matrix. We pair these values together, multiply the pairs of values, and then add to arrive at 25.

```
  R1: 1 -2   3
× C1: 1 -3   6
---------------
      1 +6 +10 = 25
```

$$\begin{bmatrix} 1 & -2 & 3 \\ 4 & 5 & -2 \end{bmatrix}\begin{bmatrix} 1 & -8 & 4 & -3 \\ -3 & 6 & 7 & 2 \\ 6 & 5 & -1 & 4 \end{bmatrix}$$

Operations with Matrices

- **Matrix Multiplication**
  - **Row 2, Column 3**
  - To find the element in row 2, column 3 of the product, we will take row 2 from the first matrix and column 3 from the second matrix. We pair these values together, multiply the pairs of values, and then add to arrive at 53.

```
  R2:  4   5   -2
× C3:  4   7   -1
----------------
       16 +35 +2 = 53
```

$$\begin{bmatrix} 1 & -2 & 3 \\ 4 & 5 & -2 \end{bmatrix} \begin{bmatrix} 1 & -8 & 4 & -3 \\ -3 & 6 & 7 & 2 \\ 6 & 5 & -1 & 4 \end{bmatrix}$$

Operations with Matrices

- **Matrix Multiplication**
  - Understanding where each number in the product comes from is helpful when you only need a specific value.
  - You don't need to multiply completely if you only want specific elements.
  - Just take the row from the first matrix and the column from the second matrix.

Operations with Matrices

- **Matrix Multiplication**
  - The process can be completed for the rest of the elements in the matrix.

		Column 1	Column 2	Column 3	Column 4
	values	[1, -3, 6]	[-0, 6, 5]	[4, 7, -1]	[-3, 2, 4]
Row 1	[1, -2, 3]	1(1) - 2(-3) + 3(6) = 1 + 6 + 10 = 25	1(-0) -2(6) + 3(5) = -0 - 12 + 15 = -5	1(4) -2(7) +3(-1) = 4 - 14 - 3 = -13	1(-3) -2(2) + 3(4) = -3 -4 + 12 = 5
Row 2	[4, 5, -2]	4(1) + 5(-3) -2(6) = 4 - 15 - 12 = -23	4(-0) + 5(6) - 2(5) = -32 + 3* - 1* = -12	4(4) + 5(7) -2(-1) = 16 + 35 + 2 = 53	4(-3) + 5(2) -2(4) = -12 + 1* - 0 = -1*

Operations with Matrices

- **Matrix Multiplication**
  - So, the final product is:

$$\begin{bmatrix} 25 & -5 & -13 & 5 \\ -23 & -12 & 53 & -10 \end{bmatrix}$$

Operations with Matrices

- **Matrix Multiplication**
  - Note that the multiplication is not defined the other way.
  - You can not multiply a 3x4 and a 2x3 matrix together because the inner dimensions aren't the same.
  - This product is undefined:

$$\begin{bmatrix} 1 & -8 & 4 & -3 \\ -3 & 6 & 7 & 2 \\ 6 & 5 & -1 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 4 & 5 & -2 \end{bmatrix}$$

Operations with Matrices

- **Matrix Multiplication**
  - Note that the multiplication is not defined the other way.
  - You can not multiply a 3x4 and a 2x3 matrix together because the inner dimensions aren't the same.
  - This product is undefined:

$$\begin{bmatrix} 1 & -8 & 4 & -3 \\ -3 & 6 & 7 & 2 \\ 6 & 5 & -1 & 4 \end{bmatrix}\begin{bmatrix} 1 & -2 & 3 \\ 4 & 5 & -2 \end{bmatrix}$$

Operations with Matrices

- **Matrix Multiplication**
  - Since the order (dimensions) of the matrices don't have to be the same, there may not be corresponding elements to multiply together.
  - Multiply the rows of the first by the columns of the second and add.

Operations with Matrices

- **There is no matrix division**
  - There is no defined process for dividing a matrix by another matrix.
  - A matrix may be divided by a scalar.

Operations with Matrices

- **Identity Matrix**
  - Square matrix
  - Ones on the main diagonal
  - Zeros everywhere else
  - Denoted by I. If a subscript is included, it is the order of the identity matrix.
  - I is the multiplicative identity for matrices
  - Any matrix times the identity matrix is the original matrix.
  - Multiplication by the identity matrix is commutative, although the order of the identity may change

Operations with Matrices

- **Identity Matrix**
  - Identity matrix of size 2 $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
  - Identity matrix of size 3 $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Operations with Matrices

- **Matrix Multiplication**
  - Matrix multiplication involves summing a product. It is appropriate where you need to multiply things together and then add. As an example, multiplying the number of units by the per unit cost will give the total cost.
  - The units on the product are found by performing unit analysis on the matrices. The labels for the product are the labels of the rows of the first matrix and the labels of the columns of the second matrix.

Operations with Matrices

- **Inverse of a Square Matrix**

  – When working in the real numbers, the equation ax=b could be solved for x by dividing both sides of the equation by a to get x=b/a, as long as a wasn't zero. It would therefore seem logical that when working with matrices, one could take the matrix equation AX=B and divide both sides by A to get X=B/A.

  – However, that won't work because ...

    There is NO matrix division!

  – Ok, you say. Subtraction was defined in terms of addition and division was defined in terms of multiplication. So, instead of dividing, I'll just multiply by the inverse. This is the way that it has to be done.

Operations with Matrices

- **Inverse of a Square Matrix**
  - So, what is the inverse of a matrix?
  - Well, in real numbers, the inverse of any real number a was the number a^{-1}, such that a times a^{-1} equalled 1.

    We knew that for a real number, the inverse of the number was the reciprocal of the number, as long as the number wasn't zero.

Operations with Matrices

- **Inverse of a Square Matrix**

  - The inverse of a square matrix A, denoted by A^{-1}, is the matrix so that the product of A and A^{-1} is the Identity matrix. The identity matrix that results will be the same size as the matrix A.

    There is a lot of similarities there between real numbers and matrices.

    $A(A^{-1}) = I$ or $A^{-1}(A) = I$

  - There are a couple of exceptions, though. First of all, A^{-1} does not mean 1/A. Remember, "There is no Matrix Division!" Secondly, A^{-1} does not mean take the reciprocal (**Covered in the next few slides**) of every element in the matrix A.

Operations with Matrices

- **Reciprocal**
  - The Reciprocal is just: 1/number
  - The reciprocal of 4 is $^1/_4$
  - The reciprocal of $^1/_4$ is 4 (back to 4 again)
  - "Reciprocal" comes from the Latin *reciprocus* meaning *returning*.
  - Every number has a reciprocal except * (1/* is undefined – NaN – divide by zero, crash, crash, crash......)
  - Multiply a Number by Its Reciprocal and You Get 1

Operations with Matrices

- **Reciprocal**
  - When multiplying a number by its reciprocal you always get 1
  - That is a way to define Reciprocal:

    What to multiply a value by to get 1!
  - In terms of multiplying it is the Inverse, so it is also called the:

    "Multiplicative Inverse".

Operations with Matrices

- **Back to the Inverse of a Square Matrix**

- **Requirements to have an Inverse**

  - The matrix must be square (same number of rows and columns).

  - The determinant (**Covered in the next slides!**) of the matrix must not be zero. This is instead of the real number not being zero to have an inverse, the determinant must not be zero to have an inverse.

  - A square matrix that has an inverse is called **invertible** or **non-singular**. A matrix that does not have an inverse is called **singular**.

  - A matrix does not have to have an inverse, but if it does, the inverse is unique.

Operations with Matrices

- **What is the Determinant of a Square matrix**
  - A determinant is a real number associated with every square matrix.
  - I have yet to find a good English definition for what a determinant is. Everything I can find either defines it in terms of a mathematical formula or suggests some of the uses of it. There's even a definition of determinant that defines it in terms of itself.
  - The determinant of a square matrix A is denoted by "det A" or | A |. Now, that last one looks like the absolute value of A, but you will have to apply context. If the vertical lines are around a matrix, it means determinant.

Operations with Matrices

- **The Determinant of a Square matrix**
  - The line below shows the two ways to write a determinant.

$$\begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix} = \det \begin{bmatrix} 3 & 1 \\ 5 & 2 \end{bmatrix}$$

Operations with Matrices

- **Properties of Determinants**
  - The determinant is a real number, it is not a matrix.
  - The determinant can be a negative number.
  - It is not associated with absolute value at all except that they both use vertical lines.
  - The determinant only exists for square matrices (2×2, 3×3, ... n×n). The determinant of a 1×1 matrix is that single value in the determinant.
  - The inverse of a matrix will exist only if the determinant is not zero.

Operations with Matrices

- **Determinants that are Zero**
  - The determinant of a matrix will be zero if :
    - An entire row is zero.
    - Two rows or columns are equal.
    - A row or column is a constant multiple of another row or column.
  - Remember, that a matrix is invertible, non-singular, if and only if the determinant is not zero. So, if the determinant is zero, the matrix is singular and does not have an inverse.

Operations with Matrices

- **Back to the Inverse of a Square Matrix – Again!!!**

- **Requirements to have an Inverse**

  – The matrix must be square (same number of rows and columns).

  – The determinant of the matrix must not be zero (determinants are covered in section 6.4). This is instead of the real number not being zero to have an inverse, the determinant must not be zero to have an inverse.

  – A square matrix that has an inverse is called **invertible** or **non-singular**. A matrix that does not have an inverse is called **singular**.

  – A matrix does not have to have an inverse, but if it does, the inverse is unique.

Operations with Matrices

- **Inverse of a Square Matrix**
  - So, either use these notes as a start point for exploring how to calculate by hand the Inverse of a Square Matrix

    OR

  - Use a calculator

    OR

  - Use your maths library

Operations with Matrices

- **So, why was it we needed an inverse?**
  - To solve a system of linear equations (more later).

    BUT for 3D CG:

  - Given a 3D point transformed via one or more matrix operations – typically those that orientate geometry with respect to the camera, the result of which is usually referred to as the view Matrix - find this points original world space ordinates.
  - Hint: Use the inverse of the view Matrxi
  - Remember the inverse of a Matrix is, simply put, the reverse of a Matrix.

Properties of Matrices

$$\mathbf{A}_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

Properties of Matrices

Property	Example
Commutativity of Addition	A + B = B + A
Associativity of Addition	A + (B + C) = (A + B) + C
Associativity of Scalar Multiplication	(cd) A = c (dA)
Scalar Identity	1A = A(1) = A
Distributive	c (A + B) = cA + cB
Distributive	(c + d) A = cA + dA
Additive Identity	A + O = O + A = A
Associativity of Multiplication	A (BC) = (AB) C
Left Distributive	A (B + C) = AB + AC
Right Distributive	(A + B) C = AC + BC
Scalar Associativity / Commutativity	c (AB) = (cA) B = A (cB) = (AB) c
Multiplicative Identity	IA = AI = A

Properties of Real Numbers that aren't Properties of Matrices

$$\mathbf{A}_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

Properties of Real Numbers that aren't Properties of Matrices

- **Commutativity of Multiplication**
  - You can not change the order of a multiplication problem and expect to get the same product.

    $AB \neq BA$

  - You must be careful when factoring common factors to make sure they are on the same side.

    $AX+BX = (A+B)X$ and $XA + XB = X(A+B)$ but $AX + XB$ doesn't factor.

Properties of Real Numbers that aren't Properties of Matrices

- **Zero Product Property**
  - Just because a product of two matrices is the zero matrix does not mean that one of them was the zero matrix.

Properties of Real Numbers that aren't Properties of Matrices

- **Multiplicative Property of Equality**
  - If A=B, then AC = BC.
    - This property is still true, but the converse is not necessarily true.
    - Just because AC = BC does not mean that A = B.
  - Because matrix multiplication is not commutative, you must be careful to either pre-multiply or post-multiply on both sides of the equation.

    That is, if A=B, then AC = BC or CA = CB, but AC≠CB.

Properties of Real Numbers that aren't Properties of Matrices

- **There is no matrix division**
  - You must multiply by the inverse of the matrix

Properties of Real Numbers that aren't Properties of Matrices

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Things to do with Matrices

$$\mathbf{A}_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

Things to do with Matrices

- **Evaluating a Function using a Matrix**
  - Consider the function f(x) = x2 - 4x + 3 and the matrix A.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

  - The initial attempt to evaluate the f(A) would be to replace every x with an A to get f(A) = A2 - 4A + 3.

    There is one slight problem, however. The constant 3 is not a matrix, and you can't add matrices and scalars together.

    So, we multiply the constant by the Identity matrix.

    f(A) = A2 - 4A + 3I.

Things to do with Matrices

- **Evaluating a Function using a Matrix**
  - Evaluate each term in the function and then add them together.

$$A^2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} * \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$$

$$-4\,A = -4 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -4 & -8 \\ -12 & -16 \end{bmatrix}$$

$$3I = 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$f(A) = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} + \begin{bmatrix} -4 & -8 \\ -12 & -16 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 3 & 9 \end{bmatrix}$$

Things to do with Matrices

- **Factoring Expressions**
  - 2X + 3X = 5X
  - AX + BX = (A+B)X
  - XA + XB = X(A+B)
  - AX + 5X = (A+5I)X
  - AX+XB does not factor

Things to do with Matrices

- **Converting Systems of Linear Equations to Matrices**
  - A system of linear equations can be written as AX=B where A is the coefficient matrix, X is a column vector containing the variables, and B is the right hand side.
  - If there are more than one system of linear equations with the same coefficient matrix, then you can expand the B matrix to have more than one column. Put each right hand side (rhs) into its own column.

Things to do with Matrices

- **Converting Systems of Linear Equations to Matrices**
  - Each equation in the system becomes a row.
  - Each variable in the system becomes a column.
  - The variables are dropped and the coefficients are placed into a matrix.
  - If the right hand side is included, it's called an augmented matrix.
  - If the right hand side isn't included, it's called a coefficient matrix.

Things to do with Matrices

- **Converting Systems of Linear Equations to Matrices**
  - The system of linear equations ...

$$
\begin{aligned}
x + y - z &= 1 \\
3x - 2y + z &= 3 \\
4x + y - 2z &= 9
\end{aligned}
$$

becomes the augmented matrix ...

$$
\begin{array}{ccc|c}
x & y & z & \text{rhs} \\
\end{array}
$$

$$
\left[\begin{array}{ccc|c}
1 & 1 & -1 & 1 \\
3 & -2 & 1 & 3 \\
4 & 1 & -2 & 9
\end{array}\right]
$$

Things to do with Matrices

- **Pivoting**
  - Totally beyond the scope of this lecture...
  - However, an easy step by step with the aid of a calculator might look like this...

Things to do with Matrices

- **Solving Systems of Linear Equations**
  - Consider the system of linear equations

```
3x + 2y - 5z =  12
 x - 3y + 2z = -13
5x -  y + 4z =  1*
```

  - Write the coefficients in an A matrix
  - Write the variables in an X matrix
  - Write the constants in a B matrix.

$$\begin{array}{ccc} \mathbf{x} & \mathbf{y} & \mathbf{z} \end{array}$$

$$\begin{bmatrix} 3 & 2 & -5 \\ 1 & -3 & 2 \\ 5 & -1 & 4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} 12 \\ -13 \\ 10 \end{bmatrix}$$

Things to do with Matrices

- **Verify that AX = B**
  - This step isn't really needed, but I wanted to show you that this thing really does work.
  - AX will be a (3×3) × (3×1) = 3×1 matrix. The B matrix is also a 3×1 matrix, so at least the dimensions work out right.
  - Here's A times X.

$$\begin{bmatrix} 3 & 2 & -5 \\ 1 & -3 & 2 \\ 5 & -1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3x + 2y - 5z \\ 1x - 3y + 2z \\ 5x - 1y + 4z \end{bmatrix}$$

Things to do with Matrices

- **This does not always work!**
  - Inverses only exist for square matrices. That means if you don't the same number of equations as variables, then you can't use this method.
  - Not every square matrix has an inverse. If the coefficient matrix A is singular (has no inverse), then there may be no solution or there may be many solutions, but we can't tell what it is.
  - Inverses are a pain to find by hand. If you have a calculator, it's not so bad, but remember that calculators don't always give you the answer you're looking for.

Things to do with Matrices

- **Back to AX = B**
  - Notice that turns out to be the left side of the system of equations. The B is the right hand side, so we have achieved equality. You can write a system of linear equations as AX = B.
  - So, if you can write a system of linear equations as AX=B where A is the coefficient matrix, X is the variable matrix, and B is the right hand side, you can find the solution to the system by $X = A^{-1} B$.

$$\begin{bmatrix} 3 & 2 & -5 \\ 1 & -3 & 2 \\ 5 & -1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3x + 2y - 5z \\ 1x - 3y + 2z \\ 5x - 1y + 4z \end{bmatrix}$$

Things to do with Matrices

- **Now get a calculator that can do Matrix maths and:**
  - Place the coefficient matrix into [A] on the calculator and the right hand side into [B].
  - If you asked the calculator to find the inverse of the coefficient matrix, it would give you this for A^{-1}

$$\begin{bmatrix} 5/44 & 3/88 & 1/8 \\ -3/44 & -37/88 & 1/8 \\ -7/44 & -13/88 & 1/8 \end{bmatrix}$$

Things to do with Matrices

– You could do that, and then multiply that by B, but it would be easier just to put the whole expression into the calculator and get the answer directly.

– Even what is shown below is more work than is necessary.

– $X = A^{-1} B = \ldots$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5/44 & 3/88 & 1/8 \\ -3/44 & -37/88 & 1/8 \\ -7/44 & -13/88 & 1/8 \end{bmatrix} \begin{bmatrix} 12 \\ -13 \\ 10 \end{bmatrix} = \begin{bmatrix} 191/88 \\ 519/88 \\ 111/88 \end{bmatrix}$$

Things to do with Matrices

– There you go:

  x = 191/00,

  y = 519/00, and

  z = 111/00.

– That would be a real pain to solve by hand!

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5/44 & 3/88 & 1/8 \\ -3/44 & -37/88 & 1/8 \\ -7/44 & -13/88 & 1/8 \end{bmatrix} \begin{bmatrix} 12 \\ -13 \\ 10 \end{bmatrix} = \begin{bmatrix} 191/88 \\ 519/88 \\ 111/88 \end{bmatrix}$$

Things to do with Matrices

- **Transform the ray from the eye through a pixel in to world space**

```
// ---------- part 4 : update view matrix ----------

// update the view matrix representing the cameras new position/orientation.
D3DXMATRIX viewMatrix;
getViewMatrix(&viewMatrix);
device->SetTransform(D3DTS_VIEW, &viewMatrix);
```

Things to do with Matrices

- **Transform the ray from the eye through a pixel in to world space**

```
// ---------- part 5 : update pick ray ----------

// Note : mouse ordinates scaled as per the ratio between the window and back buffer dimensions

// calculate the picking ray in view space
calculatePickingRay (device,
        &ray,
        stage->windowed,
        mouse->currentPosition.x0mouse->xRatio,
        mouse->currentPosition.y0mouse->yRatio);

// transform the ray to world space
D3DXMATRIX view;
D3DXMATRIX viewInverse;
D3DXMatrixInverse(&viewInverse,          *, & viewMatrix);
transformPickingRay(&ray, &viewInverse);
```

Things to do with Matrices

- **Transform the ray from the eye through a pixel in to world space**

```
// ---------- part 6 : set/unset picking mode ----------

// IF right mouse clicked THEN set picking mode

if ( mouse->rightButtonDown )
{
        ray.picking = true;
        mouse->rightButtonDown = false;
}
else
{
        ray.picking = false;
}
```

Things to do with Matrices

- **And lots more....**
  - Area of a Triangle
  - Test for Collinear Points
  - Equation of a Line
  - Cryptography
  - Solid texturing
  - Ray tracing
  - Etc.

End

$$\mathbf{A}_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$